LEBANESE AMERICAN UNIVERSITY
DEPARTMENT OF COMPUTER SCIENCE AND MATHEMATICS
MTH 304 – DIFFERENTIAL EQUATIONS
EXAM 1 – SPRING 2014

NAME: KEY. **ID #:**

THIS EXAM CONSISTS OF 9 PAGES AND 6 PROBLEMS. THE LAST PAGE MAY BE USED FOR SCRAP WORK. ANSWER THE QUESTIONS IN THE SPACES PROVIDED; IF MORE SPACE IS NEEDED, USE THE BACK OF THE PAGES. THE TIME OF THE EXAM IS 75 MINUTES.

NB. If you wish your exam graded by tomorrow (deadline for WI), tick the box ☐

Question Number	Grade
1. (28%)	
2. (14%)	
3. (13%)	
4. (12%)	
5. (18%)	
6. (15%)	
TOTAL	

1. **Identify** the following differential equations as either **separable, linear,** or **exact,** then **solve** each one **explicitly if possible.** In case the problem is **an initial-value** one, find the **particular solution.**

   a. (8%) $t\dfrac{dy}{dt}+y=t^2e^{t^3}$, $t>0$.

Divide by t: $\dfrac{dy}{dt}+\dfrac{1}{t}y=te^{t^3}$; $t>0$.

this is linear: $p(t)=\frac{1}{t}$; $q(t)=te^{t^3}$

Integrating factor $\mu(t)=e^{\int\frac{1}{t}dt}=e^{\ln t}=t$.

$\therefore y=\dfrac{1}{t}\int t.te^{t^3}dt=\dfrac{1}{t}\left[\frac{1}{3}e^{t^3}+c\right]$

$=\dfrac{1}{3t}e^{t^3}+\dfrac{c}{t}$.

   b. (8%) $e^t y\dfrac{dy}{dt}=e^{-y}+e^{-2t-y}$.

$e^t y\dfrac{dy}{dt}=e^{-y}(1+e^{-2t})$ $\therefore$ separable

$\therefore$ $ye^y dy=e^{-t}(1+e^{-2t})dt$

$\int ye^y dy=\int(e^{-t}+e^{-3t})dt=-e^{-t}-\frac{1}{3}e^{-t}+C$

$\downarrow$

$u=y \to du=dy$

$dv=e^y \to v=e^y$

$\therefore \int ye^y dy=ye^y-\int e^y dy=ye^y-e^y$

$\therefore$ Implicit family of solutions is:

$ye^y-e^y=-e^{-t}-\frac{1}{3}e^{-t}+C$.

c. **(10%)** $\left(\cos x - x\sin x + y^2\right) + 2xy\dfrac{dy}{dx} = 0;\ y(\pi) = 1.$

$$\underbrace{\left(\cos x - x\sin x + y^2\right)}_{M(x,y)} + \underbrace{2xy}_{N(x,y)}\frac{dy}{dx} = 0$$

$$\frac{\partial M}{\partial y} = 2y\ ; \quad \frac{\partial N}{\partial x} = 2y \quad \therefore \text{ exact.}$$

Solutions: $\phi(x,y) = c$, where

$$\frac{\partial \phi}{\partial x} = M \quad ; \quad \frac{\partial \phi}{\partial y} = N.$$

$$\frac{\partial \phi}{\partial y} = N \Rightarrow \phi(x,y) = \int N(x,y)\,dy = \int 2xy\,dy = xy^2 + C(x).$$

$$\frac{\partial \phi}{\partial x} = \cos x - x\sin x + \cancel{y^2} \Rightarrow \cancel{y^2} + C'(x)$$

$$\therefore C'(x) = \cos x - x\sin x$$

$$\therefore C(x) = \int(\cos x - x\sin x)\,dx$$

$$= \sin x - \underbrace{\int x\sin x\,dx}$$

$u = x \rightarrow du = dx$

$dv = \sin x \rightarrow v = -\cos x$

$$\therefore C(x) = \sin x - \left[-x\cos x + \int \cos x\,dx\right]$$

$$= \sin x + x\cos x - (\sin x)$$

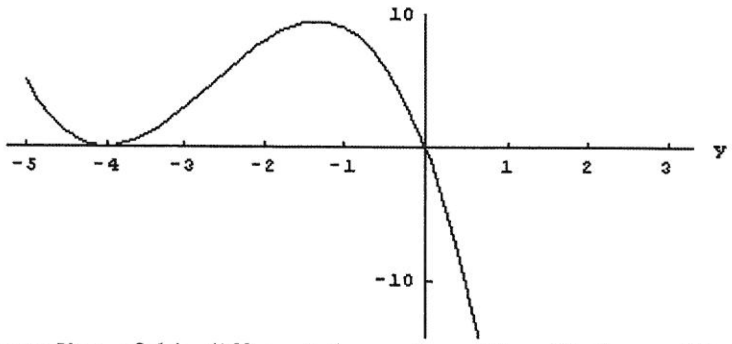
$$= x\cos x.$$

$\therefore$ Family of solutions is: $xy^2 + x\cos x = C$

But $y(\pi) = 1 \Rightarrow \pi + \pi\cos(\pi) = C \Rightarrow \pi - \pi = 0$

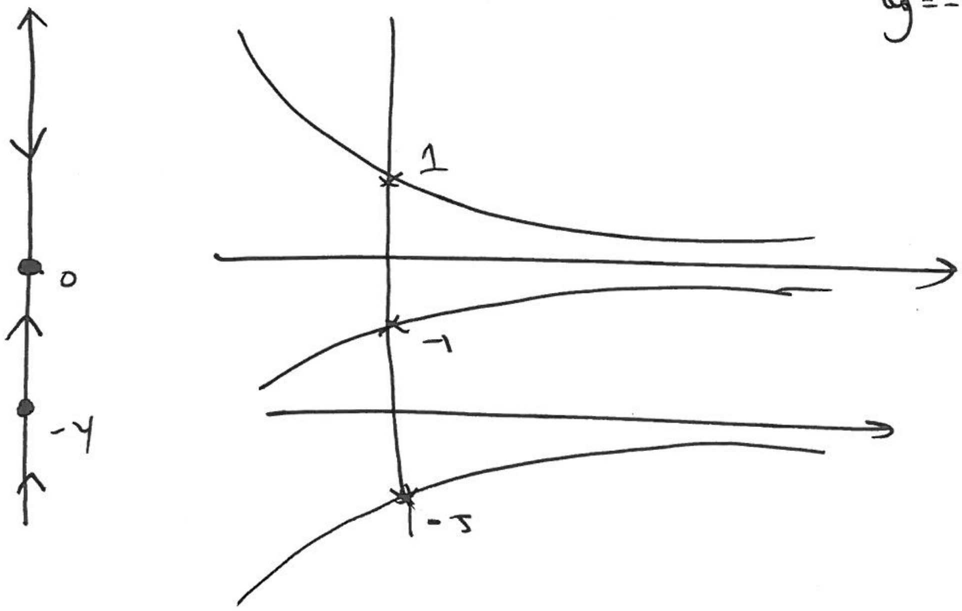
$\therefore$ family of Solutions is: $\boxed{xy^2 + x\cos x = 0.}$

2. **(14%)** Consider the **autonomous** differential equation $\dfrac{dy}{dt} = f(y)$, where the graph of $f(y)$ is given below:



Draw the **phase line** of this differential equation. **Classify its equilibrium points.**
Draw the solutions satisfying $y(t_0) = -5$, $y(t_0) = -1$ and $y(t_0) = 1$.

two eq. solutions,
$y = -4$ and $y = 0$



$y = 0$ is a sink (stable)

$y = 4$ is a node (semi-stable)

3. The air in a small rectangular room that is 20ft by 5ft by 10ft contains initially 40 liters of carbon monoxide. Air containing 0.1liters/ft^3 is blown into the room at the rate of 100ft^3/min and well mixed air flows out through a vent at the same rate.

   a. (8%) Write an initial value problem that desribes the amount of carbon monoxide in the room. **Do not solve.**

Let $Q(t)$ = amount of carbon monoxide in room $\Rightarrow$ $\underline{\underline{Q(0) = 40}}$.

input rate $= 0.1 \times 100 = 10$ liters/min.

output rate $= \dfrac{Q(t)}{20 \times 5 \times 10} \times 100 = \dfrac{Q(t)}{10}$

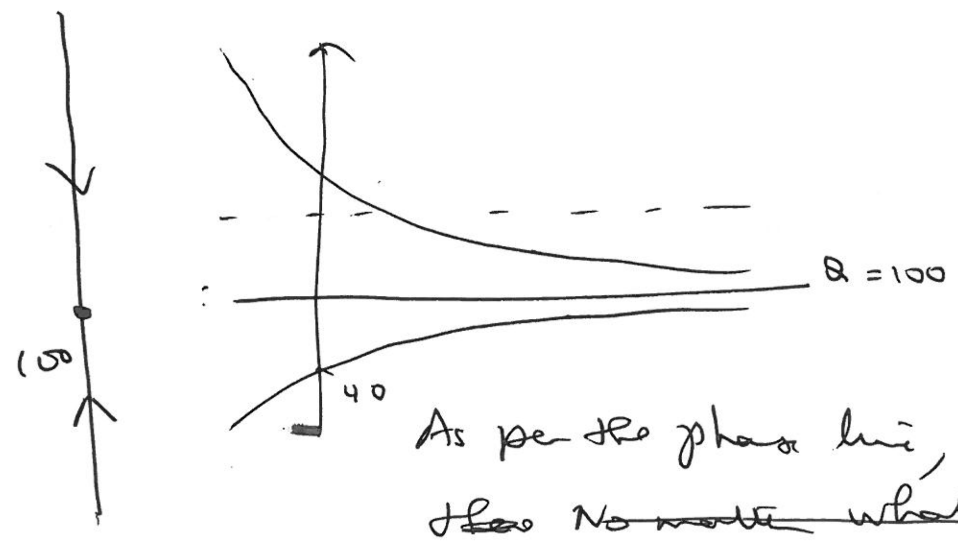
$$\therefore \frac{dQ}{dt} = 10 - \frac{Q(t)}{10} \; ; \; Q(0) = 40.$$

   b. (5%) Sketch the phase line corresponding to this ODE and **conclude from the phase line** how much carbon monoxide will be in the room after a long period of time.

this is an aut. ODE : eq. is : $\dfrac{Q(t)}{10} = 10$

$\Rightarrow Q(t) = 100$

$= \cancel{10(\ldots)}$



As per the phase line, starting ~~then No matter what initial~~ with $Q(t) = 40$, solutions tend to $Q = 100$.

5. Consider the differential equation: $\dfrac{dy}{dt} = \alpha - y^2$

   a. (9%) Discuss the existence of equilibrium solutions for various values of α (that is for what values of α, are there two ~~fixed points~~, one ~~fixed point~~, or none).

   eq. solutions eq. solu

$$\frac{dy}{dt} = 0 \Rightarrow \alpha - y^2 = 0 \Rightarrow y^2 = \alpha$$

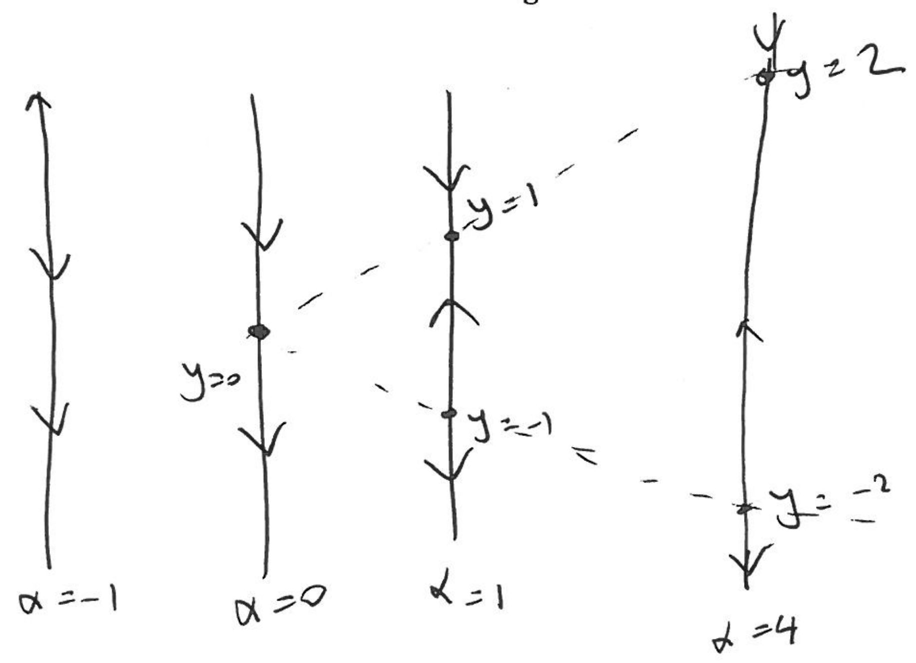
If $\alpha < 0$ $\therefore$ no eq. solutions

If $\alpha = 0 \Rightarrow y = 0$ is the only eq. solution.

If $\alpha > 0$, then there are 2: $y = \pm\sqrt{\alpha}$.

   b. (9%) Draw the phase lines **(all next to each other)** for the the following cases: $\alpha = -1, \alpha = 0, \alpha = 1, \alpha = 4$ and classify the equilibrium points.

   __Bonus:__ Try to plot a graph that join the equilibtium points for the various values of α. This is called the **bifurcation diagram.**



6. **(15%)** Consider the **4** differential equations:

a. $\dfrac{dy}{dt} = (2-y)(y+1)$

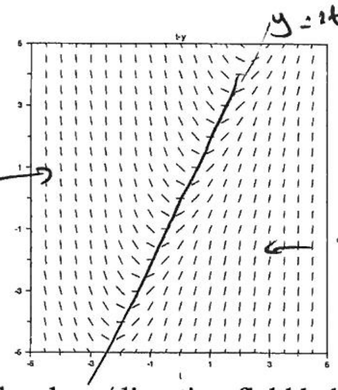
b. $\dfrac{dy}{dt} = t(y-1)$

c. $\dfrac{dy}{dt} = t^3 + 2$

d. $\dfrac{dy}{dt} = 2t - y$

Match each direction field below with exactly one ODE and justify your answer.

I. The slope/direction field below matches d because:

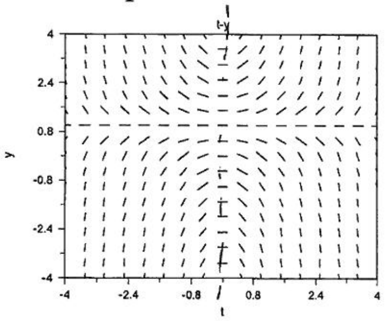


$y = 2t \Rightarrow \frac{dy}{dt} = 0$ ✓

$y > 2t$ (above line), $\frac{dy}{dt} < 0$ ✓

$y < 2t$, $\frac{dy}{dt} > 0$ ✓

II. The slope/direction field below matches b because:



Horizontal mini-tangents when $t = 0$ and when $y = 1$.

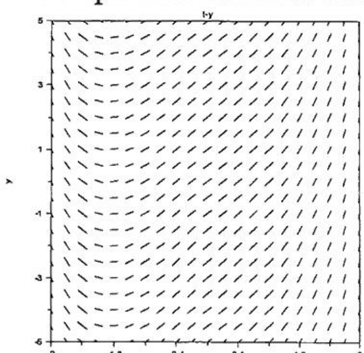
$t > 0, y > 1, \frac{dy}{dt} > 0$ ✓

$t < 0, y > 1, \frac{dy}{dt} < 0$. ✓

$t < 0, y > 1$ $\frac{dy}{dt} < 0$. ✓

$t > 0, y < 1, \frac{dy}{dt} > 0$ ✓

III. The slope/direction field below matches c because:



All the mini-tangents are parallel to each other for a fixed value of t. ($t = -\sqrt[3]{2}$).

for $t > -\sqrt[3]{2}$, $t^3 + 2 > 0 \Rightarrow$ mini tangents have positive slopes

for $t < -\sqrt[3]{2}$, mini tangents have negative slopes.

Scratch